NONLINEAR PERTURBATIONS OF LINEAR ACCRETIVE OPERATORS IN BANACH SPACES

BY

G. F. WEBB

ABSTRACT

It is shown that a strongly continuous semi-group of nonlinear nonexpansive operators can be constructed as $\lim_{n\to\infty}((I + t/nB)^{-1}(I + t/nA)^{-1})^n$ where A is a linear m-accretive operator, B is a nonlinear m-accretive operator, and B satisfies a boundedness condition relative to A.

Introduction

The objective of this paper is to investigate the perturbation of a linear m-accretive operator A by a nonlinear m-accretive operator B in a general Banach space X. Many authors have studied perturbation theory for nonlinear accretive operators in general Banach spaces and some of these are listed in our references. The basic condition used in such investigations is that B be bounded or continuous relative to A in some sense which is a nonlinear analogue to the basic condition employed in the perturbation theory of linear accretive operators. Our results relate to the relatively regular case of I. Segal [10] and T. Kato [7]. In their work it is required that B be continuously Frechet differentiable from $[D(A)]$ to X (or similar types of conditions), where $[D(A)]$ denotes the domain of A regarded as a Banach space with graph norm $\|x\| + \|Ax\|$. The basic conditions of our work here are that $\|ABx\|$ satisfy a boundedness condition and that B be continuous from $[D(A)]$ to X.

1. Definitions and theorems

In what follows $(X, \| \ \|)$ will denote a Banach space with norm $\| \ \|$. If B is an

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operator (possibly nonlinear) from X to X we will denote by $D(B)$ the domain of B and by $R(B)$ the range of B.

DEFINITION 1.1. An operator $B: X \to X$ is said to be *accretive* on X provided that

(1.1) $\| (I + \lambda B)x - (I + \lambda B)y \| \geqq \| x - y \|$ for all $x, y \in D(B)$ and $\lambda > 0$.

If in addition, $R(I + \lambda B) = X$ for all $\lambda > 0$, we say that B is *m-accretive.*

DEFINITION 1.2. By a *strongly continuous semi-group of nonexpansive operators* $U(t)$, $t \geqq 0$, on X we mean a function U from $[0, \infty) \times X$ to X such that (1) $U(t)U(s)x = U(t+s)x$ for all $t, s \geqq 0$, $x \in X$; (2) $\lim_{t \to 0^+} U(t)x = U(0)x = x$ for all $x \in X$; and (3) $\| U(t)x - U(t)y \| \leqq \| x - y \|$ for all $x, y \in X$, $t \geqq 0$. The *infinitesimal generator* of $U(t)$, $t \geqq 0$, is the function $B: x \to \lim_{t \to 0^+} (1/t)(U(t) - I)x$, defined for all x for which this limit exists.

The main theorem which we prove is the following:

THEOREM 1.1. *Suppose that A is an m-accretive linear operator on X, B is an m-accretive nonlinear operator on X, $B0 = 0$, and D is a dense subset of X such that*

(1.3) $D \subseteq D(A) \cap D(AB)$ *and* $0 \in D$;

(1.4) $(I + \lambda B)^{-1}(I + \lambda A)^{-1}(D) \subseteq D$ *for all* $\lambda > 0$;

(1.5) $\| \;\|_0$ *is a norm on D such that* $(I + \lambda A)^{-1}$ *and* $(I + \lambda B)^{-1}$ *are* $\| \;\|_0$*-nonexpansive on D for all* $\lambda > 0$;

(1.6) *There is an increasing function* $L: [0, \infty) \to [0, \infty)$ *such that for all* $x \in D$, $\| ABx \| \leqq L(\| x \|_0) \| Ax \|$.

Then, for all $x \in X$ and $t \geqq 0$

(1.7) $$U(t)x = \lim_{n \to \infty}((I + (t/n)B)^{-1}(I + (t/n)A)^{-1})^n x$$

exists, the convergence is uniform on bounded t-intervals, and $U(t)$, $t \geqq 0$, is a strongly continuous semi-group of nonlinear nonexpansive operators on X.

We will use the methods of M. Crandall and T. Liggett [4] to prove Theorem 1.1. First we establish the following lemma:

LEMMA 1.2. *Suppose that A is an m-accretive linear operator on X, B is a nonlinear m-accretive operator on X, $x \in D(A) \cap D(AB)$, and there exist positive* *constants M and* λ_0 *such that for* $0 \leqq \lambda \leqq \lambda_0$, $n = 1, 2, \cdots$, *and* $1 \leqq k \leqq n$, $((I + (\lambda/n)B)^{-1}(I + (\lambda/n)A)^{-1})^k x \in D(A)$ *and*

$$\| AB((I + (\lambda/n)B)^{-1}(I + (\lambda/n)A)^{-1})^k x \| \leqq M. \tag{1.8}$$

Then, for $0 \leqq t \leqq \lambda_0$

$$U(t)x = \lim_{n\to\infty}((I + (t/n)B)^{-1}(I + (t/n)A)^{-1})^n x \text{ exists, and} \tag{1.9}$$

$$\| U(t)x - U(t')x \| \leqq 2(\| Ax \| + \| Bx \|)| t - t'| \text{ for } 0 \leqq t \leqq t' \leqq \lambda_0. \tag{1.10}$$

PROOF. Henceforth we shall use the notation that for $\lambda > 0$, $J_\lambda = (I + \lambda B)^{-1}(I + \lambda A)^{-1}$. We first observe that $J^k_{\lambda/n}x \in D(A)$ implies $J^k_{\lambda/n}x \in D(AB)$, since $D(A)$ is a linear subspace of X and

$$(I - (I + (\lambda/n)B)^{-1})(I + (\lambda/n)A)^{-1}J^{k-1}_{\lambda/n}x = (\lambda/n)BJ^k_{\lambda/n}x.$$

Next we establish the identity, for $0 < \mu \leqq \lambda$ and $J_\lambda x \in D(A)$

$$J_\lambda x = J_\mu((\mu/\lambda)x + ((\lambda - \mu)/\lambda)J_\lambda x - \mu(\lambda - \mu)ABJ_\lambda x). \tag{1.11}$$

We shall use from [4] (Lemma 1.2–(iv)) the fact that

$$(I + \mu B)(I + \lambda B)^{-1} = (\mu/\lambda)I + ((\lambda - \mu)/\lambda)(I + \lambda B)^{-1}.$$

Set

$$\begin{aligned}
&(I + \mu A)(I + \mu B)J_\lambda x \\
&= (I + \mu A)((\mu/\lambda)(I + \lambda A)^{-1}x + ((\lambda - \mu)/\lambda)J_\lambda x) \\
&= (\mu/\lambda)((\mu/\lambda)x + ((\lambda - \mu)/\lambda)(I + \lambda A)^{-1}x) + ((\lambda - \mu)/\lambda)(I + \mu A)J_\lambda x \\
&= (\mu/\lambda)((\mu/\lambda)x + ((\lambda - \mu)/\lambda)(x - \lambda A(I + \lambda A)^{-1}x)) + ((\lambda - \mu)/\lambda)(I + \mu A)J_\lambda x \\
&= (\mu/\lambda)x - (\mu/\lambda)(\lambda - \mu)A(I + \lambda A)^{-1}x + ((\lambda - \mu)/\lambda)J_\lambda x + (\mu/\lambda)(\lambda - \mu)AJ_\lambda x \\
&= (\mu/\lambda)x + ((\lambda - \mu)/\lambda)J_\lambda x - \mu(\lambda - \mu)ABJ_\lambda x.
\end{aligned}$$

We next establish the inequality, for $x \in D(A) \cap D(AB)$, $\lambda \geqq 0$, and $k \geqq 1$,

$$\begin{aligned}
\| J^k_\lambda x - x \| &\leqq \sum_{i=1}^{k} \| J_\lambda x - x \| \\
&\leqq \sum_{i=1}^{k} \| x - (I + \lambda A)(I + \lambda B)x \| \\
&\leqq k\lambda(\| Ax \| + \| Bx \|) + k\lambda^2 \| ABx \|.
\end{aligned} \tag{1.12}$$

Now let $0 \leqq \mu \leqq \lambda$, $k, j \geqq 0$, and define $a_{k,j} = \| J^j_\mu x - J^k_\lambda x \|$. Next let $\alpha = \mu/\lambda$, and $\beta = 1 - \alpha$. If $n \geqq j \geqq 1$ and $m \geqq k \geqq 1$, then from (1.11)

$$
\begin{aligned}
a_{k,j} &= \| J_\mu^j x - J_\mu((\mu/\lambda)J_\lambda^{k-1}x + ((\lambda-\mu)/\lambda)J_\lambda^k x - \mu(\lambda-\mu)ABJ_\lambda^k x)\| \\
&\leqq (\mu/\lambda)\| J_\mu^{j-1}x - J_\lambda^{k-1}x\| + ((\lambda-\mu)/\lambda)\| J_\mu^{j-1}x - J_\lambda^k x\| \\
&\quad + \mu(\lambda-\mu)\| ABJ_\lambda^k x \\
&\leqq \alpha a_{k-1,j-1} + \beta a_{k,j-1} + \mu(\lambda-\mu)M.
\end{aligned}
$$

By Lemma A in the Appendix of [6] we obtain, for $m \leqq n$,

$$
\begin{aligned}
a_{m,n} \leqq & \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} \| J_\lambda^{m-j}x - x\| \\
& + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} \| J_\mu^{n-j}x - x\| \\
& + \left(\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \alpha^i \beta^{j-i} \right) \mu(\lambda-\mu)M
\end{aligned}
$$

and by Lemma 2.1 of [6] and by (1.12) above we obtain

$$
\begin{aligned}
a_{m,n} &\leqq \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} ((m-j)\lambda(\| Ax\| + \| Bx\|) + (m-j)\lambda^2 M) \\
&\quad + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} ((n-j)\mu(\| Ax\| + \| Bx\|) + (n-j)\lambda^2 M) \\
&\quad + n\mu(\lambda-\mu)M \\
&\leqq ((n\alpha - m)^2 + n\alpha\beta)^{\frac{1}{2}}(\lambda(\| Ax\| + \| Bx\|) + \lambda^2 M) \\
&\quad + ((m\beta/\alpha^2) + ((m\beta)/\alpha + m - n)^2)^{\frac{1}{2}}(\mu(\| Ax\| + \| Bx\|) + \mu^2 M) \\
&\quad + n\mu(\lambda-\mu)M \\
&= ((n\mu - \lambda m)^2 + n\mu(\lambda-\mu))^{\frac{1}{2}}(\| Ax\| + \| Bx\| + \lambda M) \\
&\quad + (m\lambda(\lambda-\mu) + (m\lambda - n\mu)^2)^{\frac{1}{2}}(\| Ax\| + \| Bx\| + \mu M) + n\mu(\lambda-\mu)M.
\end{aligned}
$$

Taking $\mu = t/n$ and $\lambda = t/m$ where $0 \leqq t \leqq \lambda_0$ and $n \geqq m$, we obtain

$$
\begin{aligned}
&\| J_{t/n}^n x - J_{t/m}^m x\| \\
\text{(1.13)} \qquad &\leqq t(1/m - 1/n)^{\frac{1}{2}}(2\| Ax\| + 2\| Bx\| + (t/m)M + (t/n)M) \\
&\quad + t^2(1/m - 1/n)M
\end{aligned}
$$

and so $\lim_{n\to\infty} J_{t/n}^n x$ exists. Taking $n = m$, $0 \leqq t \leqq t' \leqq \lambda_0$, $\mu = t/n$, $\lambda = t'/n$, and letting $n \to \infty$, we obtain (1.10).

Proof of Theorem 1.1. First observe that $\| (I + \lambda B)^{-1}x\| \leqq \| x\|$ for all

$x \in X$ and $\| (I + \lambda B)^{-1} x \|_0 \leqq \| x \|_0$ for all $x \in D$, since $B0 = 0$. Next, if $x \in D$ and $0 < \lambda < 1/L(\| x \|_0)$, then

$$\begin{aligned} \| AJ_\lambda x \| &\leqq \| A(I + \lambda A)^{-1} x \| + \lambda \| ABJ_\lambda x \| \\ &\leqq \| Ax \| + \lambda L(\| J_\lambda x \|_0) \| AJ_\lambda x \| \\ &\leqq \| Ax \| + \lambda L(\| x \|_0) \| AJ_\lambda x \|, \end{aligned}$$

which implies that

$$\| AJ_\lambda x \| \leqq (1 - \lambda L(\| x \|_0))^{-1} \| Ax \|.$$

Thus, if $x \in D$, $\lambda \geqq 0$, $n > \lambda L(\| x \|_0)$, and $1 \leqq k \leqq n$, then

$$\| AJ^k_{\lambda/n} x \| \leqq (1 - (\lambda/n) L(\| x \|_0))^{-k} \| Ax \|,$$

which implies that

$$(1.14) \qquad \begin{aligned} \| ABJ^k_{\lambda/n} x \| &\leqq L(\| J^k_{\lambda/n} \|_0) \| AJ^k_{\lambda/n} x \| \\ &\leqq L(\| x \|_0) (1 - (\lambda/n) L(\| x \|_0))^{-k} \| Ax \|. \end{aligned}$$

By Lemma 1.2, if $x \in D$ and $\lambda > 0$, then $\lim_{n\to\infty} J^n_{\lambda/n} x$ exists. Since D is dense in X and the J_λ's are nonexpansive on X, $\lim_{n\to\infty} J^n_{\lambda/n} x$ exists for all $x \in X$ and all $\lambda > 0$.

Define $U(t)x = \lim_{n\to\infty} J^n_{t/n} x$ for $x \in X$, $t \geqq 0$. Obviously the nonlinear operators $U(t)$, $t \geqq 0$, are nonexpansive as the strong limits of nonexpansive operators. To see that $U(t)$, $t \geqq 0$, is strongly continuous we first note that if $x \in D$, then (1.10) yields the strong continuity of $U(t)x$, $t \geqq 0$. Then the facts that $U(t)$, $t \geqq 0$, are nonexpansive and $D(A) \cap D(AB)$ is dense in X yield the strong continuity for all $x \in X$. Finally, we verify that $U(t + s) = U(t)\, U(s)$, $t, s \geqq 0$. Following [4] we have for $t \geqq 0$, m, n positive integers, and $x \in X$,

$$(U(t))^m x = \lim_{n\to\infty} (J^n_{t/n})^m x = \lim_{n\to\infty} (J^m_{t/n})^n x$$

which implies

$$U(mt) = \lim_{n\to\infty} J^n_{mt/n} = \lim_{k\to\infty} J^{mk}_{mt/mk} = \lim_{k\to\infty} (J^m_{t/k})^k = U(t)^m.$$

Let l, k, m, n be positive integers and then

$$\begin{aligned} U(l/k + m/n) &= U((ln + mk)/kn) = (U(1/kn))^{ln+km} \\ &= (U(1/kn))^{ln} (U(1/kn))^{km} = U(l/k)\, U(m/n). \end{aligned}$$

Thus, $U(t + s)x = U(t)U(s)x$ where t, s are rational numbers and so by the strong continuity and nonexpansive property of $U(t)$, $t \geqq 0$, $U(t + s)x = U(t)\, U(s)x$ for

for all $t, s \geqq 0$, $x \in X$. Therefore, $U(t)$, $t \geqq 0$, is a strongly continuous semi-group of nonexpansive operators on X and the proof of Theorem 1.1 is complete.

We next investigate the relationship of the semi-group $U(t)$, $t \geqq 0$ (constructed in Theorem 1.1) to $A + B$. With additional hypothesis on A and B, we can show that the infinitesimal generator of $U(t)$, $t \geqq 0$, extends $-(A + B)$. We first require the following.

DEFINITION 1.3. Suppose that A is an operator on X, B is an operator on X, and $D \subseteq D(A) \cap D(B)$. We define B to be *A-continuous* on D to mean that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in D, $x_0 \in D$, $x_n \to x_0$, and $Ax_n \to Ax_0$, then $Bx_n \to Bx_0$.

THEOREM 1.4. *Suppose that the hypothesis of Theorem* 1.1 *is satisfied. Suppose further that B is A-continuous on D. Let C be the infinitesimal generator of $U(t)$, $t \geqq 0$. Then, $D \subseteq D(C)$ and $-Cx = (A + B)x$ for all $x \in D$.*

PROOF. We first prove that for any $x \in X$

$$\lim_{t \to 0} \| (I + (t/n)A)^{-k}x - x \| = 0 \tag{1.15}$$

uniformly for all $n = 1, 2, \cdots$, and $1 \leqq k \leqq n$. But since $D(A)$ is dense in X and $(I + (t/n)A)^{-k}$ is a contraction, it suffices to prove the result for $x \in D(A)$, in which case it follows from

$$\begin{aligned} &\| (I + (t/n)A)^{-k}x - x \| \\ &\leqq \sum_{i=1}^{k} \| (I + (t/n)A)^{-i}x - (I + (t/n)A)^{-(i-1)}x \| \\ &\leqq k \| (I + (t/n)A)^{-1}x - x \| \\ &\leqq (kt/n) \| Ax \| \leqq t \| Ax \|. \end{aligned}$$

Next we note the following identity (which one proves by induction): For $n = 1, 2, \cdots$, $1 \leqq i \leqq n$, $x \in X$, and $t \geqq 0$,

$$J_{t/n}^{i}x = (I + (t/n)A)^{-i}x - (t/n) \sum_{k=1}^{i} (I + (t/n)A)^{-(i-k)}BJ_{t/n}^{k}x. \tag{1.16}$$

We will use these facts below to show that if $x \in D$ then,

$$\lim_{t \to 0} \| (1/t)(U(t)x - x) + (A + B)x \| = 0. \tag{1.17}$$

It suffices to show that for any decreasing sequence of positive numbers $\{t_m\}_{m=1}^{\infty}$ converging to 0

$$\lim_{m \to \infty} \| (1/t_m)(U(t_m)x - x) + (A + B)x \| = 0. \tag{1.18}$$

Let $x \in D, \{t_m\}_{m=1}^{\infty}$ decrease to 0, and $\varepsilon > 0$. By virtue of (1.13) and (1.14), we have that for all $m = 1,2,\cdots$, $n = 1,2,\cdots$,

$$\| U(t_m)x - J^n_{t_m/n}x \| \leqq t_m(1/n)^{\frac{1}{2}}(2\| Ax \| + 2\| Bx \| + (t_m/n)M) + t_m^2(1/n)M$$

where M depends only on x and the sequence $\{t_m\}_{m=1}^{\infty}$. Thus, there exists an n such that

$$(1/t_m)\| U(t_m)x - J^n_{t_m/n}x \| < \varepsilon/4 \text{ for all } m = 1,2,\cdots. \tag{1.19}$$

Then, for all $m = 1,2,\cdots$,

$$\begin{aligned}
&\| (1/t_m)(U(t_m)x - x) + (A + B)x \| \\
&\quad < \varepsilon/4 + \| (1/t_m)(J^n_{t_m/n}x - x) + (A + B)x \| \\
&= \varepsilon/4 + \| (1/t_m)((I + (t_m/n)A)^{-n}x - x \\
&\quad - (t_m/n) \sum_{i=1}^{n} (I + (t_m/n)A)^{-(n-i)}BJ^i_{t_m/n}x) + (A + B)x \| \\
&\leqq \varepsilon/4 + \| (1/t_m)(I + (t_m/n)A)^{-n}x - x) + Ax \| \\
&\quad + (1/n) \sum_{i=1}^{n} \| (I + (t_m/n)A)^{-(n-i)} BJ^i_{t_m/n}x - Bx \|
\end{aligned}$$

$$\leqq \varepsilon/4 + \| (1/t_m)(I + (t_m/n)A)^{-n}x - x) + Ax \| \tag{1.20}$$

$$+ (1/n) \sum_{i=1}^{n} \| BJ^i_{t_m/n}x - Bx \| \tag{1.21}$$

$$+ (1/n) \sum_{i=1}^{n} \| (I + (t_m/n)A)^{-(n-i)}Bx - Bx \|. \tag{1.22}$$

Then, there exists m_1 such that the second term of (1.20) $< \varepsilon/4$ for all $m \geqq m_1$ by virtue of (1.15) and the following inequality:

$$\| (1/t_m)(1 + (t_m/n)A)^{-n}x - x) + Ax\|$$

$$= \| (1/t_m)(t_m/n) \sum_{i=1}^{n} A(I + (t_m/n)A)^{-i}x - Ax \| \tag{1.23}$$

$$\leqq (1/n) \sum_{i=1}^{n} \| (I + (t_m/n)A)^{-i}Ax - Ax \|.$$

Next, there exists m_2 such that (1.21) $< \varepsilon/4$ for all $m \geqq m_2$ by virtue of the facts that B is A continuous, (1.12), (1.14), (1.23), and the following inequalities:

$$\| AJ^i_{t_m/n}x - Ax \|$$

$$= \| A((I + (t_m/n)A)^{-i}x - (t_m/n) \sum_{k=1}^{i} (I + (t_m/n)A)^{-(i-k)}BJ^k_{t_m/n}x) - Ax \|$$

$$\leqq \| (I + (t_m/n)A)^{-i}Ax - Ax \|$$

$$+ (t_m/n) \sum_{k=1}^{i} \| (I + (t_m/n)A)^{-(i-k)}ABJ^k_{t_m/n}x \|$$

$$\leqq \| (I + (t_m/n)A)^{-i}Ax - Ax \|$$

$$+ t_m L(\| x \|_0)(1 - (t_m/n)L(\| x \|_0))^{-i} \| Ax \|.$$

Finally, there exists m_3 such that (1.22) $< \varepsilon/4$ for all $m \geqq m_3$ by virtue of (1.15). Thus, for $m \geqq \max\{m_1, m_2, m_3\}$

$$\| (1/t_m)(U(t_m)x - x) + (A + B)x \| < \varepsilon$$

which implies (1.18) and the theorem is proved.

THEOREM 1.5. *Suppose that the hypothesis of Theorem* 1.1 *is satisfied. Suppose further that* $A + B$ *is m-accretive. Then, for all* $x \in X$ *and* $t \geqq 0$,

$$\lim_{n\to\infty}((I + (t/n)B)^{-1}(I + (t/n)A)^{-1})^n x = \lim_{n\to\infty}(I + (t/n)(A + B))^{-n}x. \tag{1.24}$$

PROOF. We first observe that $\lim_{n\to\infty}(I + (t/n)(A + B))^{-n}x$ exists for all $x \in X$, $t \geqq 0$, by virtue of Theorem I in [4]. Next observe that for $z \in D$, $\lambda > 0$,

$$(I + \lambda(A + B))z = (I + \lambda A)(I + \lambda B)z - \lambda^2 ABz.$$

Then, recalling (1.4), we have for all $x \in D$, $\lambda > 0$,

$$\| (I + \lambda(A + B))^{-1}x - (I + \lambda B)^{-1}(I + \lambda A)^{-1}x \|$$

$$\leqq \| x - (I + \lambda(A + B))(I + \lambda B)^{-1}(I + \lambda A)^{-1}x \|$$

$$= \lambda^2 \| AB(I + \lambda B)^{-1}(I + \lambda A)^{-1}x \|.$$

Now let $x \in D$, $t > 0$, $n \geqq 1$, and we have that

$$\| (I + (t/n)(A + B))^{-n}x - ((I + (t/n)B)^{-1}(I + (t/n)A)^{-1})^n x \|$$

$$\leqq \sum_{i=1}^{n} (t/n)^2 \| AB((I + (t/n)B)^{-1}(I + (t/n)A)^{-1})^i x \|.$$

By (1.14) we see that (1.24) is true for all $x \in D$. But since D is dense in X, (1.24) holds for all $x \in X$.

Our final result is due to A. Pazy, whose helpful suggestions in revising this paper we gratefully acknowledge.

THEOREM 1.6. *Let A be a densely defined closed linear m-accretive operator on X. Let B be accretive everywhere defined and continuous on X. Then, for all $x \in X$ and $t \geqq 0$,*

$$\lim_{n\to\infty}((I + (t/n)B)^{-1}(I + (t/n)A)^{-1})^n x = \lim_{n\to\infty}(I + (t/n)(A + B))^{-n}x. \tag{1.25}$$

PROOF. The existence of $\lim_{n\to\infty}(I + (t/n)(A + B))^{-n}x$ follows from [13], where it was proved that $A + B$ is m-accretive, and Theorem I of [4]. If we can prove that

$$\lim_{\lambda\to 0}(1/\lambda)(I - (I + \lambda B)^{-1}(I + \lambda A)^{-1})x = (A + B)x \tag{1.26}$$

for all $x \in D(A)$, then (1.25) will follow from Corollary 4.3 of H. Brezis and A. Pazy [2]. But (1.26) follows from

$$\begin{aligned} &\| (1/\lambda)(I - (I + \lambda B)^{-1}(I + \lambda A)^{-1})x - (A + B)x \| \\ &\leqq \| (1/\lambda)((I + \lambda A)^{-1}x - (I + \lambda B)^{-1}(I + \lambda A)^{-1}x) - Bx \| \\ &\quad + \| (1/\lambda)(x - (I + \lambda A)^{-1}x) - Ax \| \\ &= \| B(I + \lambda B)^{-1}(I + \lambda A)^{-1}x - Bx \| + \| (I + \lambda A)^{-1}Ax - Ax \|. \end{aligned}$$

and the continuity of B.

2. Examples

We conclude with two examples illustrating the hypothesis of Theorem 1.1.

EXAMPLE 2.1. Let $X = L^p_{[0,\infty]}$ where $1 \leqq p < \infty$. Let $Ax = -x'$, $D(A) = \{x \in X : x' \in X\}$. Let b be a continuously differentiable nondecreasing real-valued function defined everywhere on the real line R such that $b(0) = 0$ and define $B: X \to X$ by $(Bx)(s) = b(x(s))$ for all $x \in X$ such that $Bx \in X$. Let $A_1 x = -x'$, $D(A_1) = \{x \in C_{[0,\infty]} : x' \in C_{[0,\infty]}\}$, where $C_{[0,\infty]}$ is the space of bounded uniformly continuous real-valued functions on $[0, \infty)$. It is well-known that A is a linear m-accretive operator on X, B is a nonlinear m-accretive operator on X, and A_1 is a linear m-accretive operator on $C_{[0,\infty]}$. Let $D = D(A) \cap D(A_1) \cap D(AB)$ and it is obvious that D is dense in X. We shall verify below the conditions (1.3), (1.4), (1.5), and (1.6).

We show first that $(I + \lambda A)^{-1}(D(A) \cap D(A_1)) \subseteq D(A) \cap D(A_1)$ for all $\lambda > 0$.

Suppose $x \in D(A) \cap D(A_1)$ and let $y = (I + \lambda A)^{-1}x$. Obviously, $y \in D(A)$. Furthermore, y satisfies uniquely $y(t) - \lambda y'(t) = x(t)$ where $y \in X$. Then,

$$y(t) = \exp(t/\lambda)\,(y(0) - (1/\lambda) \int_0^t \exp(-s/\lambda)x(s)ds),$$

$$y(0) = (1/\lambda) \int_0^\infty \exp(-s/\lambda)x(s)ds$$

(since if $y(0) \neq (1/\lambda) \int_0^t \exp(-s/\lambda)x(s)\,ds$, then $y \notin X$). But y is also in $C_{[0,\infty]}$ and so y satisfies $y(t) - \lambda y'(t) = x(t)$ uniquely for $y \in C_{[0,\infty]}$. Hence, $(I + \lambda A)^{-1}x = (I + \lambda A_1)^{-1}x$ and so $y \in D(A_1)$ and $(I + \lambda A)^{-1}$ and $(I + \lambda A_1)^{-1}$ agree on $D(A) \cap D(A_1)$.

We show next that $(I + \lambda B)^{-1}(D(A) \cap D(A_1)) \subseteq D(A) \cap D(A_1)$. Let $x \in D(A) \cap D(A_1)$ and let $y = (I + \lambda B)^{-1}x$. Then, $y = (I + \lambda b)^{-1}(x)$ and

$$y' = x'/(1 + \lambda b')\,((I + \lambda B)^{-1}x).$$

Since b is nondecreasing and continuously differentiable, $y' \in X$ and $y' \in C_{[0,\infty]}$. Thus, $(I + \lambda B)^{-1}x \in D(A) \cap D(A_1)$. Note also that

$$\| (I + \lambda B)^{-1}x \|_{C[0,\infty]} \leqq \| x \|_{C[0,\infty]},\ x \in D(A) \cap D(A_1)$$

because $(I + \lambda b)^{-1}$ is Lipschitz continuous with Lipschitz constant 1 and $(I + \lambda b)^{-1}0 = 0$. Thus, (1.4) is satisfied, since if $x \in D(A) \cap D(A_1)$, then

$$B(I + \lambda B)^{-1}(I + \lambda A)^{-1}x = (1/\lambda)(I - (I + \lambda B)^{-1})\,(I + \lambda A)^{-1}x$$

and so $(I + \lambda B)^{-1}(I + \lambda A)^{-1}x \in D(A) \cap D(A_1) \cap D(AB)$. Obviously (1.3) is satisfied and the remarks above show that (1.5) is satisfied for $\| \ \|_0 = \| \ \|_{C[0,\infty]}$.

It remains to show (1.6) Let $x \in D$ and then

$$\| ABx \|^p_{L^p_{[0,\infty]}} = \int_0^\infty | (b'(x) \cdot x')(s)|^p ds$$
$$\leqq \| b'(x) \|^p_{C_{[0,\infty]}} \| Ax \|^p_{L^p_{[0,\infty]}}$$

which implies that

$$\| ABx \|_X \leqq L(\| x \|_0)\ \| Ax \|_X$$

where $L\colon [0, \infty) \to [0, \infty)$ is an increasing function such that $(b'(r)) \leqq L(|r|)$ for all real numbers r. Thus all the conditions of Theorem 1.1 are verified. For an example of b in Example 2.1, one could take $p = 1$ and $b(s) = s^3$ and we note that in this case B is not everywhere defined, not continuous, and $D(A)$ is not contained in $D(B)$.

EXAMPLE 2.2. Let $X = L^1_{(-\infty,\infty)}$, let $Ax = -x''$, $D(A) = \{x \in X : x'' \in X\}$, let b be a twice continuously differentiable nondecreasing real-valued function defined on the real line R such that $b(0) = 0$ and define $B: X \to X$ by $(Bx)(s) = b(x(s))$ for all $x \in X$ such that $Bx \in X$. Let $C_{[-\infty,\infty]}$ be the space of bounded and uniformly continuous realvalued functions on $(-\infty,\infty)$. It is well-known that A is a linear m-accretive operator on X and B is a nonlinear m-accretive operator on X and also on $C_{[-\infty,\infty]}$ where its domain is taken as all $x \in C_{[-\infty,\infty]}$ such that $Bx \in C_{[-\infty,\infty]}$. Let $D = D(A) \cap D(AB) \cap C_0^2(R) = C_0^2(R)$, where $C_0^2(R)$ denotes the space of twice continuously differentiable functions on R with compact support. It is clear that D is dense in X. We shall show that the conditions of Theorem 1.1 are all satisfied.

We first show that $(I + \lambda A)^{-1}(C_0^2(R)) \subseteq C_0^2(R)$ for all $\lambda > 0$. Suppose $x \in C_0^2(R)$ and let $y = (I + \lambda A)^{-1}x$. Thus y satisfies uniquely $y(t) - \lambda y''(t) = x(t)$ where $y \in X$. Then,

$$\begin{aligned} y(t) &= (1/2\sqrt{\lambda})(\exp(t\sqrt{\lambda})\int_t^\infty \exp(-s/\sqrt{\lambda})x(s)\,ds \\ &\quad - \exp(-t/\sqrt{\lambda})\int_t^\infty \exp(s/\sqrt{\lambda})x(s)ds) \\ &= (1/2\sqrt{\lambda})(\exp(-t/\sqrt{\lambda})\int_{-\infty}^t \exp(s/\sqrt{\lambda})x(s)ds \\ &\quad - \exp(t/\sqrt{\lambda})\int_{-\infty}^t \exp(-s/\sqrt{\lambda})x(s)ds) \end{aligned}$$

since y satisfies the above differential equation and $y \in X$ (note that $x \in C_0^2(R)$ implies the integrals above exist). From the expressions for y above, it is clear that y is twice continuously differentiable with compact support. Hence, $(I + \lambda A)^{-1}$ leaves $C_0^2(R)$ invariant. Note further that

$$\|(I + \lambda A)^{-1}x\|_{C_{[-\infty,\infty]}} \leqq \|x\|_{C_{[-\infty,\infty]}}, \quad x \in C_0^2(R),$$

since $A_1 x = -x''$, $D(A_1) = \{x \in C_{[-\infty,\infty]} : x'' \in C_{[-\infty,\infty]}\}$ is an m-accretive linear operator on $C_{[-\infty,\infty]}$ and $(I + \lambda A)^{-1}x = (I + \lambda A_1)^{-1}x$ for all $x \in C_0^2(R)$ (by the uniqueness of the solutions to $y - \lambda y'' = x$).

Now observe that $(I + \lambda B)^{-1}(C_0^2(R)) \subseteq C_0^2(R)$, since $(I + \lambda B)^{-1} = (I + \lambda b)^{-1}(x)$ and b is twice continuously differentiable and nondecreasing. Note also that $\|(I + \lambda B)^{-1}x\|_{C_{[-\infty,\infty]}} \leqq \|x\|_{C_{[-\infty,\infty]}}$. Therefore, (1.4) is satisfied, since if $x \in C_0^2(R) \subseteq D(A)$,

$$B(I + \lambda B)^{-1}(I + \lambda A)^{-1}x = (1/\lambda)(I - (I + \lambda B)^{-1}(I + \lambda A)^{-1}x$$

and so $(I+\lambda B)^{-1}(I+\lambda A)^{-1}x \in D(A) \cap D(AB) \cap C_0^2(R) = D$. Obviously (1.3) is satisfied and we have established above that condition (1.5) is satisfied for $\| \ \|_0 = \| \ \|_{C[-\infty,\infty]}$.

Finally, we show (1.6) is satisfied. Let $x \in D$. Then, by integration by parts,

$$\| ABx \|_{L^1_{(-\infty\ \infty)}} = \int_{-\infty}^{\infty} \left| (b''(x) \cdot x'^2 + b'(x) \cdot x'')(s) \right| ds$$

$$\leqq \| b''(x) \|_{C[-\infty,\infty]} \left(- \int_{-\infty}^{\infty} x''(s)x(s)ds \right) + \| b'(x) \|_{C[-\infty,\infty]} \int_{-\infty}^{\infty} | x''(s) | \, ds$$

$$\leqq \| b''(x) \|_{C[-\infty,\infty]} \| x \|_{C[-\infty,\infty]} \| x'' \|_{L^1_{(-\infty,\infty)}} + \| b'(x) \|_{C[-\infty,\infty]} \| x'' \|_{L^1_{(-\infty,\infty)}}$$

$$= (\| b''(x) \|_0 \| x \|_0 + \| b'(x) \|_0) \| Ax \|_X$$

$$\leqq (L_1(\| x \|_0) \| x \|_0 + L_1(\| x \|_0)) \| Ax \|_X$$

where $L_1 : [0,\infty) \to [0,\infty)$ is an increasing function such that $b'(r) \leqq L_1(|r|)$ and $b''(r) \leqq L_1(|r|)$ for all real numbers r. Thus, (1.6) is satisfied with $L(r) = L(r) \cdot r + L_1(r)$ and so all the conditions of Theorem 1.1 are verified.

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Vanderbilt University
Nashville, Tennessee 37235